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Some recent results on James and von Neumann-Jordan constants

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Among various geometric constants of a Banach space X the von Neumann-Jordan constant $C_{NJ}(X)$ and James constant $J(X)$ have been treated most widely. Since Kato, Maligranda and Takahashi [3] showed a relation (inequalities) between these constants, several authors have improved their result. Recently covering all the previous results, the present authors showed that the quite simple inequality $C_{NJ}(X) \leq J(X)$ holds for all Banach spaces X . In this expository short note we shall present a concise description of these developments.

Let X be a real Banach space with $\dim X \geq 2$. The closed unit ball and unit sphere of X are denoted by B_X and S_X , respectively. The *James (non-square) constant* of X is

$$J(X) = \sup \{ \min(\|x + y\|, \|x - y\|) : x, y \in S_X \}, \quad (1)$$

and the *von Neumann-Jordan constant* of X is

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}. \quad (2)$$

The modified von Neumann-Jordan constant $C'_{NJ}(X)$ is defined by taking the supremum over all $x, y \in S_X$ in (2). Recall that a Banach space X is *uniformly non-square* provided $J(X) < 2$ or equivalently $C_{NJ}(X) < 2$.

1. The relation between $J(X)$ and $C_{NJ}(X)$ was first investigated by Kato-Maligranda-Takahashi [3] who proved that

$$\frac{J(X)^2}{2} \leq C_{NJ}(X) \leq \frac{J(X)^2}{1 + (J(X) - 1)^2}, \quad (3)$$

where all the terms coincide if $J(X) = 2$, i.e., X is not uniformly non-square. The first inequality attains equality with many uniformly non-square spaces, while the second is strict for all uniformly non-square spaces.

2. In 2004 Nikolova-Persson-Zachariades [5] improved the second inequality as

$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4} \left[\sqrt{J(X)^2 - 4J(X) + 8} - 2 \right] \quad (4)$$

(see also Maligranda et al. [4], Takahashi [7]). Maligranda formulated the following conjecture: For all Banach space X

$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1. \quad (5)$$

3. In 2008 Alonso et al. [1] proved that

$$C_{NJ}(X) \leq 2 \left[1 + J(X) - \sqrt{2J(X)} \right], \quad (6)$$

which was obtained by combining the inequalities

$$C'_{NJ}(X) \leq J(X) \quad (7)$$

and

$$C_{NJ}(X) \leq 2 \left[1 + C'_{NJ}(X) - \sqrt{2C'_{NJ}(X)} \right]. \quad (8)$$

Note that

$$2 \left[1 + J(X) - \sqrt{2J(X)} \right] \leq J(X)^2/4 + 1, \quad (9)$$

where equality holds only when $J(X) = 2$. This indicates that the inequality (6) is sharper than (5), a fortiori, (4). Thus they answered Maligranda's conjecture (5) affirmatively.

4. A slight improvement of the inequality (6) was obtained by Wang-Pang [10] in 2009:

$$C_{NJ}(X) \leq J(X) + \sqrt{J(X) - 1} \left[\sqrt{1 + (1 - \sqrt{J(X) - 1})^2} - 1 \right]. \quad (10)$$

To prove (10) they used the inequalities

$$\rho_X(1) \leq \sqrt{J(X) - 1} \quad (11)$$

and

$$C'_{NJ}(X) \leq J(X), \quad (12)$$

where $\rho_X(\tau)$ is the modulus of smoothness of X .

5. Covering all the previous results, we proved in [9] that the inequality

$$C_{NJ}(X) \leq J(X) \quad (13)$$

holds true for all Banach spaces X , where equality holds only when X is not uniformly non-square. This answered affirmatively the question mentioned in Alonso et al. [1, Question 1]. More precisely, we first improved the inequality (11) by Wang-Pang [10] as

$$\rho_X(1) \leq 2 \left\{ 1 - \frac{1}{J(X)} \right\} \leq \sqrt{J(X) - 1}. \quad (14)$$

The first inequality of (14) is equivalent to

$$\frac{2}{2 - \rho_X(1)} \leq J(X). \quad (15)$$

Secondly we showed that

$$C_{NJ}(X) \leq 1 + \rho_X(1) \left[\sqrt{\{1 - \rho_X(1)\}^2 + 1} - \{1 - \rho_X(1)\} \right]. \quad (16)$$

It is easy to see that

$$1 + \rho_X(1) \left[\sqrt{\{1 - \rho_X(1)\}^2 + 1} - \{1 - \rho_X(1)\} \right] \leq \frac{2}{2 - \rho_X(1)}.$$

Therefore by combining (16) and (15), we obtained that $C_{NJ}(X) \leq J(X)$.

6. Recently we considered the inequality $C_{NJ}(X) \leq J(X)$ by another approach with the modified von Neumann-Jordan constant $C'_{NJ}(X)$ and obtained another proof of the above inequality and some other results, which will appear elsewhere.

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